# PHY 711: ANALYTICAL DYNAMICS

## Additional Practice Problems II

### Problem 1

A particle of mass $m$ starts from the north pole and moves with constant speed $v$ along a longitude to reach the south pole of Earth. We can consider Earth to be a sphere of mass $M$ and radius $R$ which rotates freely in space. Earth's initial angular velocity when the particle starts its trip is $\omega$. Show that when the particle reaches the south pole, the rotation of Earth will have been retarded by an angle

$$\alpha = \frac{\pi\omega R}{v}\left[1 - \left(\frac{2M}{2M+5m}\right)^{\frac{1}{2}}\right]$$

### Problem 2

A small ball of radius $R$ can roll on the inner surface of a bowl in the shape of a paraboloid placed on the ground as shown. The paraboloid is described by $z = \frac{1}{2}\lambda r^2$, where $r^2 = x^2 + y^2$, and $\lambda$ is a positive constant. Write down the Lagrangian describing the motion of the ball. (*Hint*: Use cylindrical coordinates for which $ds^2 = dz^2 + dr^2 + r^2 d\varphi^2$. Moment of inertia for a sphere $= (2/5)MR^2$)

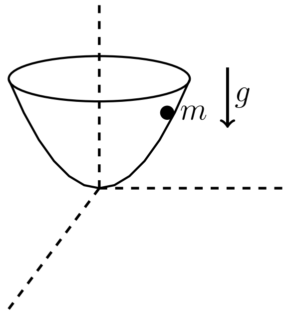
### Problem 3

A disk of mass $M$ and radius $R$ can roll on a horizontal plane along the $x$-axis. (Ignore motion along the other directions.) It is attracted to a fixed point $P$ with a force $F = -ks$ where $s$ is the distance of the center of the disk from $P$ (See figure). Obtain the Lagrangian and the equations of motion. The disk can oscillate (in a rolling motion) as shown. Obtain the frequency of this oscillation. (Moment of inertia of a disk for rotations around its axis is $\frac{1}{2}MR^2$. (*Hint: Obtain the potential energy corresponding to $F$ and then use the geometry to relate $s$ to $x$. To obtain the potential energy, it is easier to consider the force split into its components along the horizontal and vertical axes.*)

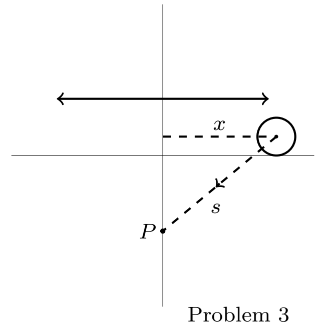
### Problem 4

The motion of a class of asteroids shepherded by the planet Jupiter can be described by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}m\,\left[(\dot{Q}_1 - \omega Q_2)^2 + (\dot{Q}_2 + \omega Q_1)^2\right] + \frac{\mu_1}{\sqrt{(Q_1 - \alpha_2 R)^2 + Q_2^2}} + \frac{\mu_2}{\sqrt{(Q_1 + \alpha_1 R)^2 + Q_2^2}}$$




Problem 2




Problem 3

where $Q_1, Q_2$ are the position coordinates in the plane of the orbit and $\omega$ is the orbital angular velocity of Jupiter, taken as constant. $\mu_1, \mu_2, \alpha_1, \alpha_2, R$ are all constants. Obtain the Hamiltonian and the canonical equations of motion. (*There are two groups of asteroids, called the Trojans and the Greeks, one of which is* $60^o$ *ahead of Jupiter and the other is* $60^o$ *behind Jupiter in its orbit which are kept in the same relative position by the combined gravitational force of the Sun and Jupiter and the centrifugal effect of their revolution around the Sun. This is a classic problem of celestial mechanics. The given Lagrangian describes the dynamics of these asteroids.*)

## Problem 5

Consider a hoop of radius $R$ placed vertically on a horizontal flat frictionless surface. It can rotate around the vertical axis. A bead of mass $m$ is free to slide on the hoop, see figure.

a) Obtain the Lagrangian of the system for the bead and the hoop.

b) Obtain the equations of motion.

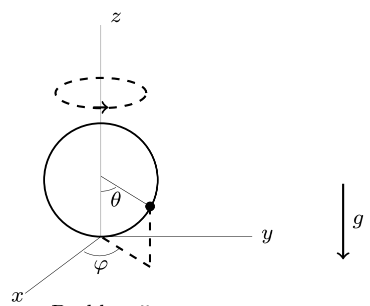
c) Identify the canonical momenta and work out the Hamiltonian.

d) Obtain the canonical equations of motion.

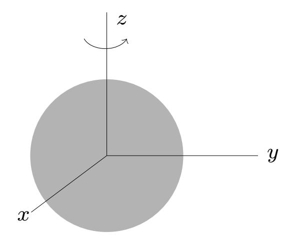
(*The rotation of the hoop is not externally controlled, it is part of the dynamics. The moment of inertia for a hoop of mass* $M$ *for rotation around one of its diameters is* $I = \frac{1}{2}MR^2$.)

## Problem 6

Consider a uniform circular disc of small thickness, mass $M$ and radius $R$. Calculate its moment of inertia for rotations around one of the diameters; i.e., calculate $I_{33} = I_{zz}$ for the disc shown in figure. (This is like a coin spinning on its edge.) (*You may encounter integrals like* $\int d\theta \sin^4\theta$ *or* $\int d\theta \cos^4\theta$. *You can simplify them using trigonometric identities expressing* $\sin^2\theta$ *and* $\cos^2\theta$ *in terms of* $\cos 2\theta$.)



Problem 5



Problem 6

## Problem 7

Consider a rail bent into a semicircular shape, of radius $R$, and placed on the ground as shown in figure. A disk of mass $M$ and radius $a$ can roll without slipping on the inside of this rail. (The motion is effectively one dimensional.)

a) Obtain the Lagrangian for the motion of the disk. (Moment of inertia of disk is $\frac{1}{2}Ma^2$.)

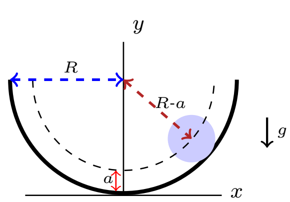
b) Obtain the Hamiltonian and the canonical equations of motion.

(*Caution*: The distance traced out by the center of mass of the disk is different from the distance it rolls because $a$ is not negligible.)

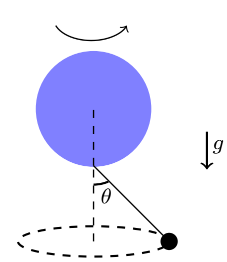
c) Determine the frequency for small oscillations around the equilibrium point if $a = R/3$.

## Problem 8

A solid spherical ball of mass $M$ and radius $R$ floats at some height above the ground. It is kept at a fixed height by some means not relevant to the problem. (You may think of it as a spherical approximation to a hot air balloon.) From the bottom of the ball is hung a pendulum with a bob of mass $m$, using a thin rigid rod of negligible mass and fixed length $l$. The mass $m$ can move in $\theta$ and in the azimuthal angle. The ball itself can rotate around a vertical axis. (We will ignore other types of motion for the ball.) Obtain the Lagrangian and the equations of motion for the system. (Keep in mind that there can be independent rotational motion for the ball and the bob of the pendulum.)



Problem 7



Problem 8

**Problem 9**

A rigid body with principal moments of inertia $I_1$ and $I_2 = I_3$ moves under the action of a torque

$$\tau_1 = 0, \qquad \tau_2 = A\,\cos\omega t, \qquad \tau_3 = A\,\sin\omega t$$

Find the angular velocities $\Omega_1$, $\Omega_2$, $\Omega_3$ as functions of time, for arbitrary initial conditions.

**Problem 10**

Show that the following change of variables is a canonical transformation.

$$Q = \log\left(\frac{1}{q}\sin p\right), \qquad\qquad P = q\cot p$$